Modeling Oscillatory Components:

div(BMO) *and Homogeneous Besov Spaces*

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Outline

1. Variational Image Decomposition.
2. Motivation.
   - Mumford-Shah and Rudin-Osher-Fatemi models.
   - Y. Meyer modeling oscillatory components with

   $$G = \text{div}(L^{\infty}),\ F = \text{div}(BMO),\ E = \dot{B}^{-1}_{\infty,\infty}.$$

   - Vese-Osher's approximation of Meyer G-model.
   - Osher-Sole-Vese model with $\dot{H}^{-1}$.
3. Modeling Oscillatory components with div(BMO).
4. Modeling oscillatory components with Besov spaces.
5. Numerical results.

Publications

This presentation consists of materials from these papers:

1. T. Le and L. Vese, *Image decomposition using total variation and div*(bmo), Multiscale Modeling and Simulation, SIAM Interdisciplinary Journal, vol.4, num. 2, pp. 390-423, June 2005.
2. J. Garnett, T. Le, and L. Vese, *Image decompositions using bounded variation and homogeneous Besov spaces*, UCLA CAM Report 05-57, Oct. 2005.

Variational image decomposition

Let f be periodic with the fundamental domain $\Omega = [-\frac{1}{2}, \frac{1}{2}]^2 \subset \mathbb{R}^2$. Denote L^2 for $L^2(\Omega)$, etc.
A variational method for decomposing f into $u + v$,

- u is piecewise smooth,
- v is oscillatory or noise,

can be given by an energy minimization problem

$$\inf_{(u,v) \in X_1 \times X_2} \left\{ \mathcal{K}(u, v) = F_1(u) + \lambda F_2(v) : f = u + v \right\}, \text{ where}$$

- $F_1, F_2 \geq 0$ are functionals on spaces of functions or distributions X_1, X_2, respectively. $\lambda > 0$.
- A good model for $\mathcal{K}$ is given by a choice of X_1 and X_2 so that $F_1(u) << F_2(u)$, and $F_1(v) >> F_2(v)$.

Mumford-Shah (1989)

$$\inf_{(u,v)\in SBV\times L^2}\left\{\left[\int_{\Omega\setminus J_u}|\nabla u|^2dx+\alpha\mathcal{H}^1(J_u)\right]+\lambda\|v\|_{L^2}^2,\ f=u+v\right\}.$$

- $f\in L^\infty\subset L^2$ is split into $u\in SBV$, a piecewise-smooth function with its discontinuity set J_u composed of a union of curves, and $v=f-u\in L^2$ representing noise or texture.
- $\mathcal{H}^1$ denotes the 1-dimensional Hausdorff measure,
- $\alpha,\lambda>0$ are tuning parameters.

With the above notations, the first two terms compose $F_1(u)$ (non-convex), while the third term makes $F_2(v)$.

Rudin-Osher-Fatemi (1992)

$$\inf_{(u,v)\in BV\times L^2}\left\{\int|\nabla u|\,dx+\lambda\|v\|_{L^2}^2,\quad f=u+v\right\},$$

- $\int|\nabla u|\,dx$ denotes $|u|_{BV}$, $\lambda>0$ is a tuning parameter.
- $f\in L^2$ is split into $u\in BV$, a piecewise-smooth function and $v=f-u\in L^2$ representing noise or texture.

With the above notation, $F_1(u)=|u|_{BV}$, and $F_2(v)=\|v\|_{L^2}^2$.
• **Loss of Intensity:** Let $f=\chi_D$ be the characteristic function of a disk D centered at 0 of radius R. The minimizer (u,v) of ROF is given by:

$$u=(1-\frac{1}{\lambda R})\chi_D,\ v=\frac{1}{\lambda R}\chi_D.$$

ROF (cont.)

• Replacing $\|v\|_{L^2}^2$ with $\|v\|_{L^1}$ was proposed by Cheon, Paranjpye, Vese and Osher as a Summer project, and further analysis by Chan and Esedoglu, Esedoglu and Vixie, Allard, among others.

Remark: Oscillatory components do not have small norms in L^2 or L^1.

• To overcome these drawbacks, we have to relax the conditions on u or $v = f - u$. One way is to use a non-convex regularization on u (like in Mumford-Shah model), that is weaker than BV. Another way is to use weaker norms than the L^2 norm.

• Here we choose to keep BV, and consider weaker norms than L^2.

Meyer models (2001)

• Mumford-Gidas (2001) also shows that natural images are drawn from probability distributions supported by generalized functions.
In 2001, Y. Meyer proposed (weaker norms)

$$\inf_{(u,v)\in BV\times X_2}\left\{|u|_{BV}+\lambda\|v\|_{X_2},\quad f=u+v\right\}.$$

Here X_2 is either G, F, or E.
• The space $G=$ div(L^∞) consists of distributions v which can be written as

$$v=\mathrm{div}(\vec{g}),\quad \vec{g}=(g_1,g_2)\in\left(L^\infty\right)^2,$$ with

$$\|v\|_G=\inf\left\{\left\|\sqrt{(g_1)^2+(g_2)^2}\right\|_{L^\infty}\ :\ v=\mathrm{div}(\vec{g}),\ \vec{g}\in\left(L^\infty\right)^2\right\}.$$

Meyer (cont.)

- The space $F = \text{div}(BMO)$ consists of distributions v which can be written as

$$v = \text{div}(\vec{g}), \quad \vec{g} = (g_1, g_2) \in (BMO)^2, \text{ with}$$

$$\|v\|_F = \inf\left\{\|g_1\|_{BMO} + \|g_2\|_{BMO} : \; v = \text{div}(\vec{g}), \; \vec{g} \in (BMO)^2\right\}.$$

We say that f belongs to BMO, if

$$\|f\|_{BMO} = \sup_{Q \subset \Omega} \frac{1}{|Q|} \int_Q |f - f_Q| < \infty,$$

where $Q \subset \Omega$ is a square (with sides parallel with the axis). Here $f_Q = |Q|^{-1} \int_Q f(x,y)$ denotes the mean value of f over the square Q.

Meyer (cont.)

- We say a generalized function v belongs to the space E if it can be written as $v = \Delta g$, such that

$$\sup_{|y|>0} \frac{\|g(.+y) - 2g(.) + g(.-y)\|_{L^\infty}}{|y|} < \infty.$$

- Both $G = \text{div}(L^\infty)$ and $F = \text{div}(BMO)$ (as defined previously) consist of first order derivatives of vector fields in L^∞ and BMO, respectively.
- E (as defined above) consists of second order derivatives of functions satisfying the Zygmund condition.
- In $\mathbb{R}^2$: $BV \subset L^2 \subset G \subset F \subset E$.
- **Difficulty:** How to solve these models in practice.

Approximating Meyer G-model

Vese-Osher (2003): model oscillatory components as first order derivatives of vector fields in L^p, for $1 \le p < \infty$.

$$\inf_{u,\vec{g}} \left\{ |u|_{BV} + \mu \|f - u - \text{div}(\vec{g})\|_{L^2}^2 + \lambda \left\| \sqrt{g_1^2 + g_2^2} \right\|_{L^p} \right\}.$$

- $f \in L^2$ is decomposed into $u + v + r$, such that $u \in BV$, $v = \text{div}(\vec{g}) \in \text{div}(L^p)$, and $r = f - u - v \in L^2$ is a residual which is negligible numerically for large μ.
- As $\mu,\ p \to \infty$, this model approaches Meyer G-model.
- Other motivating work on the G space includes Aujol et al, Aubert and Aujol, S. Osher and O. Scherzer, among others.

Osher-Sole-Vese (2003)

- In 2003, S. Osher, A. Sole, and L. Vese model oscillatory components as $v = \Delta g$, where $g \in \dot{H}^1$. I.e. $v \in \dot{H}^{-1}$.

$$\inf_{u,v} \left\{ |u|_{BV} + \lambda \left\| \nabla(\Delta^{-1} v) \right\|_{L^2}^2 ,\ f = u + v \right\} .$$

- L. Lieu and L. Vese (2005) recently consider modeling oscillatory components as $v \in H^s$, $s \in \mathbb{R}^-$,

$$\inf_{u,v} \left\{ |u|_{BV} + \lambda \int_\Omega \left| (1 + |\xi|^2)^{s/2} \hat{v}(\xi) \right|^2 \, d\xi,\ f = u + v \right\} .$$

Modeling Oscillatory components with div(BMO)

- We consider a strictly convex variational problem:

$$\inf_{u,\vec{g}} \Big\{ \mathcal{F}_1(u,\vec{g}) = |u|_{BV} + \mu \|f - u - \text{div}(\vec{g})\|_{L^2}^2 + \lambda \left[\|g_1\|_{BMO} + \|g_2\|_{BMO} \right] \Big\}$$

- A more isotropic problem: $\vec{g} = \nabla \cdot g$, i.e. $v = \Delta g$,

$$\inf_{u,g} \Big\{ \mathcal{F}_2(u,g) = |u|_{BV} + \mu \|f - u - \Delta g\|_{L^2}^2 + \lambda \left[\|g_x\|_{BMO} + \|g_y\|_{BMO} \right] \Big\}$$

Here, $f = u + v + r$, where $r \in L^2$ is a residual. As $\mu \to \infty$, these models approach Meyer F-model.

Minimizing $\mathcal{F}_1(u, \vec{g})$

$$\begin{aligned}
\mathcal{F}_1(u,\vec{g}) &= |u|_{BV} + \mu \|f - u - \text{div}(\vec{g})\|_{L^2}^2 \\
&\quad + \lambda \Big[\|g_1\|_{BMO} + \|g_2\|_{BMO} \Big] \\
&= \int_\Omega |\nabla u| \; + \mu \int_\Omega |f - u - \partial_x g_1 - \partial_y g_2|^2 \\
&\quad + \lambda \Big[\frac{1}{|B_1|} \int_\Omega |g_1 - g_{1,B_1}| H(\phi_1) + \frac{1}{|B_2|} \int_\Omega |g_2 - g_{2,B_2}| H(\phi_2) \Big],
\end{aligned}$$

- ϕ_i is the level set of B_i, H is the heaviside function, and
- $g_{i,B_i} = \frac{\int_\Omega g_i H(\phi_i)}{\int_\Omega H(\phi_i)}$,
- B_i maximizes $\|g_i\|_{BMO}$.

Minimizing $\mathcal{F}_1(u, \vec{g})$ (cont.)

Keeping B_1 and B_2 fixed for one iteration, and minimizing $\mathcal{F}_1(u, \vec{g})$ with respect to its variables, we obtain

$$-div\left(\frac{\nabla u}{|\nabla u|}\right) - 2\mu(f - u - \partial_x g_1 - \partial_y g_2) = 0,$$

$$2\mu\partial_x(f - u - \partial_x g_1 - \partial_y g_2)$$
$$+\frac{\lambda H(\phi_1)}{|B_1|}\left[\frac{g_1 - g_{1,B_1}}{|g_1 - g_{1,B_1}|} - \frac{1}{|B_1|}\int_\Omega \frac{g_1 - g_{1,B_1}}{|g_1 - g_{1,B_1}|} H(\phi_1)\right] = 0,$$

$$2\mu\partial_y(f - u - \partial_x g_1 - \partial_y g_2)$$
$$+\frac{\lambda H(\phi_2)}{|B_2|}\left[\frac{g_2 - g_{2,B_2}}{|g_2 - g_{2,B_2}|} - \frac{1}{|B_2|}\int_\Omega \frac{g_2 - g_{2,B_2}}{|g_2 - g_{2,B_2}|} H(\phi_2)\right] = 0,$$

Minimizing $\mathcal{F}_2(u,g)$

$$\mathcal{F}_2(u,g) = \int |\nabla u| \, + \mu \int_\Omega |f - u - \Delta g|^2$$
$$+ \lambda \Big[\frac{1}{|B_1|} \int_\Omega |g_x - g_{x,B_1}| H_\epsilon(\phi_1)$$
$$+ \frac{1}{|B_2|} \int_\Omega |g_y - g_{y,B_2}| H_\epsilon(\phi_2) \Big]$$

- H_ϵ is a smooth approximation of the Heaviside function H, and
- the unknown sets B_1 and B_2 maximize the BMO norms of $g_1 = g_x$ and of $g_2 = g_y$,
- ϕ_i is a level set of B_i.

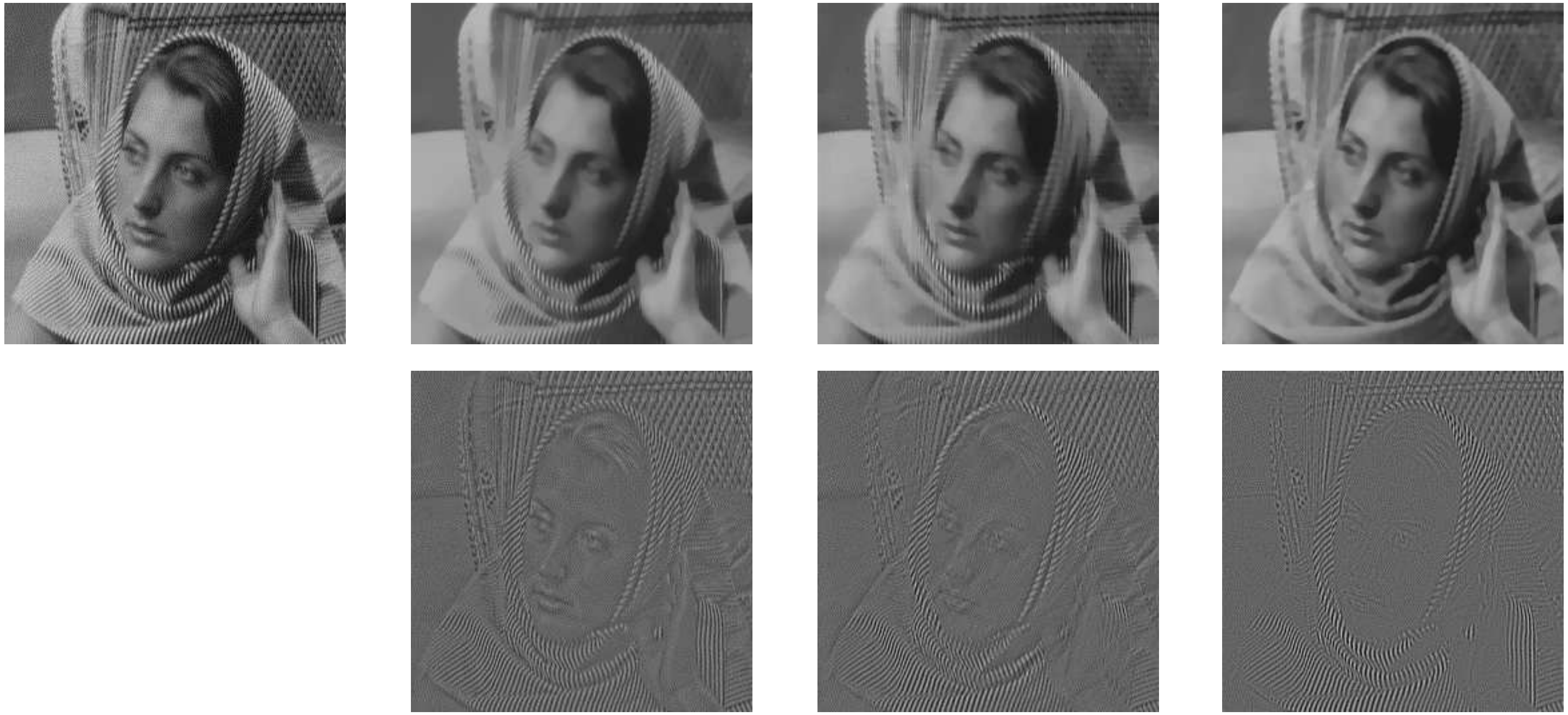
Minimizing $\mathcal{F}_2(u,g)$ (cont.)

For fixed B_1 and B_2 (at one iteration), minimizing $\mathcal{F}_2(u,g)$ w.r.t. u and g, we obtain the Euler-Lagrange equations:

$$
\begin{aligned}
& - div\left(\frac{\nabla u}{|\nabla u|}\right) - 2\mu\,(f-u-\Delta g) = 0, \\
& - 2\mu\Delta(f-u-\Delta g) - \frac{\lambda}{|B_1|}\left[\partial_x\left(\frac{g_x - g_{x,B_1}}{|g_x - g_{x,B_1}|}H_\epsilon(\phi_1)\right)\right] \\
& + \frac{\lambda}{|B_1|^2}\left(\int \frac{g_x - g_{x,B_1}}{|g_x - g_{x,B_1}|}H(\phi_1)\right)\partial_x H_\epsilon(\phi_1) \\
& - \frac{\lambda}{|B_2|}\left[\partial_y\left(\frac{g_y - g_{y,B_2}}{|g_y - g_{y,B_2}|}H_\epsilon(\phi_2)\right)\right] \\
& + \frac{\lambda}{|B_2|^2}\left(\int \frac{g_y - g_{y,B_2}}{|g_y - g_{y,B_2}|}H_\epsilon(\phi_2)\right)\partial_y H_\epsilon(\phi_2) = 0,
\end{aligned}
$$

Numerical results and comparisons

ROF, $v \in L^2$ $\quad$ $v = \mathrm{div}(\vec{g})$ $\quad$ $v = \Delta g$



Remark: The case $v = \Delta g$ is more isotropic, and better capturing repeated patterns.

Homogeneous Besov spaces

Consider the Poisson and the Gaussian kernels,

$$P_t(x) \qquad = (e^{-2\pi t|\xi|})^\vee(x) = c_n \frac{t}{(t^2+|x|^2)^{(n+1)/2}}.$$

$$W_t(x) \qquad = (e^{-2\pi t|\xi|^2})^\vee(x) = a_n t^{-n/2} e^{-\frac{\pi|x|^2}{2t}}.$$

For each $g \in L^p$. Let $w(x,t) = P_t * g(x)$, and $h(x,t) = W_t * g(x)$. We have

- $\frac{\partial^2 w}{\partial t^2} = -\Delta w$ (the wave equation).
- $\frac{\partial h}{\partial t} = \Delta h$ (the heat equation).

Besov spaces (cont.)

Let $\alpha \in \mathbb{R}$, $k, m \in \mathbb{N}_0$ s.t. $k > \alpha$ and $m > \alpha/2$, $1 \le p \le \infty$. We say $g \in \dot{B}^{\alpha}_{p,q}$, if

$$\begin{aligned}
\|g\|_{\dot{B}^{\alpha}_{p,q}} &= \left(\int \left| t^{k-\alpha} \left\| \frac{\partial^k P_t}{\partial t^k} * g \right\|_{L^p} \right|^q \frac{dt}{t} \right)^{1/q} \\
&\approx \left(\int \left| t^{m-\alpha/2} \left\| \frac{\partial^m W_t}{\partial t^m} * g \right\|_{L^p} \right|^q \frac{dt}{t} \right)^{1/q} < \infty, \;\; q < \infty. \\
\|g\|_{\dot{B}^{\alpha}_{p,\infty}} &= \sup_{t>0} \left\{ t^{k-\alpha} \left\| \frac{\partial^k P_t}{\partial t^k} * g \right\|_{L^p} \right\} \\
&\approx \sup_{t>0} \left\{ t^{m-\alpha/2} \left\| \frac{\partial^m W_t}{\partial t^m} * g \right\|_{L^p} \right\} < \infty, \;\; q = \infty.
\end{aligned}$$

Some properties of $\dot{B}^{\alpha}_{p,q}$

- Denote $I_s v = (-\Delta)^{s/2}(v) = ((2\pi|\xi|)^s \hat{v}(\xi))^{\vee}$, We have

$$I_s : \dot{B}^{\alpha}_{p,q} \to \dot{B}^{\alpha-s}_{p,q}, \text{ isometrically (injectively).}$$

- Define $\tau_\delta f(x) = f(\delta x),\ \delta > 0$. We have

$$\|\tau_\delta f\|_{L^p(\mathbb{R}^n)} = \delta^{-\frac{n}{p}} \|f\|_{L^p(\mathbb{R}^n)}, \text{ and}$$

$$\|\tau_\delta f\|_{\dot{B}^{\alpha}_{p,q}(\mathbb{R}^n)} = \delta^{-\frac{n}{p}+\alpha} \|f\|_{\dot{B}^{\alpha}_{p,q}(\mathbb{R}^n)}, \text{ for } \alpha < 0,\ 1 \le p,q < \infty.$$

- The following embedding holds,

$$\dot{B}^{\alpha_1}_{p,q_1} \subset \dot{B}^{\alpha_2}_{p,q_2},$$

if either $\alpha_2 \le \alpha_1$, or $\alpha_1 = \alpha_2$ and $1 \le q_1 \le q_2 \le \infty$.

Modeling oscillatory components with homogeneous Besov spaces

- Meyer E-model corresponds to modeling

$$u \in BV, \text{ and } v = \Delta g, \; g \in \dot{B}^{1}_{\infty,\infty}. \text{ I.e. } v \in \dot{B}^{-1}_{\infty,\infty}.$$

- (Joint work with **J. Garnett** and **L. Vese**) We consider decomposing $f = u + v$, such that

$$u \in BV, \text{ and } v = \Delta g \in \dot{B}^{\alpha-2}_{p,\infty}, \; g \in \dot{B}^{\alpha}_{p,\infty}, \; 0 < \alpha < 2, 1 \leq p \leq \infty,$$

with the minimization problems

- $\inf_{u,g} \left\{ \mathcal{J}_a(u,g) = |u|_{BV} + \mu \|f - u - \Delta g\|^2_{L^2} + \lambda \|g\|_{\dot{B}^{\alpha}_{p,\infty}} \right\}$
- $\inf_{u} \left\{ \mathcal{J}_e(u) = |u|_{BV} + \lambda \|f - u\|_{\dot{B}^{\alpha-2}_{p,\infty}} \right\}$

- Aujol & Chambolle: Meyer E-model (wavelets).

Minimizing $\mathcal{J}_a$, $p < \infty$

$$\mathcal{J}_a(u,g) = |u|_{BV} + \mu \, \|f - u - \Delta g\|^2_{L^2} + \lambda \|g\|_{\dot{B}^{\alpha}_{p,\infty}},$$
$$= \int_\Omega |\nabla u| + \mu \int_\Omega |f - u - \Delta g|^2 + \lambda \sup_{t>0} \|K_t^\alpha * g\|_{L^p},$$

where $K_t^\alpha = t^{2-\alpha}\frac{\partial^2 P_t}{\partial t^2}$ or $K_t^\alpha = t^{1-\alpha/2}\frac{\partial W_t}{\partial t}$.
In practice, we consider only a discrete set

$$\left\{ t_i = 2.5\tau^i : \ \tau = 0.9, \ i = 1, ..., N = 150 \right\}.$$

These t_i's are chosen so that discretely $P_{t_1}(x)$ is a constant and $P_{t_N}(x)$ approximates the Dirac delta function.

Algorithm

- Given an initial guess (u_0, g_0).
- Compute $\bar{t}_0 =$ argmax$_{t\in\{t_1,\dots,t_N\}} \|K_t^\alpha * g_0\|_{L^p}$.
- Suppose $(u_n, g_n, \bar{t}_n)$ is known. Compute (u_{n+1}, g_{n+1}) via

$$\left(\frac{\partial \mathcal{J}_a}{\partial u} =\right),\ 0 = -\nabla\cdot\left(\frac{\nabla u}{|\nabla u|}\right) - 2\mu(f - u - \Delta g)$$

$$\left(\frac{\partial \mathcal{J}_a}{\partial g} =\right),\ 0 = -2\mu\Delta(f - u - \Delta g) +$$

$$\lambda \left\|K_{\bar{t}}^\alpha * g\right\|_{L^p}^{1-p} K_{\bar{t}_n}^\alpha * \left(\left|K_{\bar{t}_n}^\alpha * g\right|^{p-2} K_{\bar{t}}^\alpha * g\right)$$

- Suppose $\bar{t}_n = t_k$. Compute $\bar{t}_{n+1} =$ argmax$_{t\in\{t_{k-1},t_k,t_{k+1}\}} \|K_t^\alpha * g_{n+1}\|_{L^p}$. Continue...

Minimizing $\mathcal{J}_a,\ p=\infty$

$$\mathcal{J}_a(u,g) = |u|_{BV} + \mu \, \|f-u-\Delta g\|_{L^2}^2 + \lambda \|g\|_{\dot{B}^{\alpha}_{\infty,\infty}},$$
$$= \int_\Omega |\nabla u| + \mu \int_\Omega |f-u-\Delta g|^2 + \lambda \sup_{t>0, h\in L^1} \frac{\langle K_t^\alpha * g, h\rangle}{\|h\|_{L^1}}.$$

• **Algorithm:** The steps are the same as in the previous case, but now at each iteration we need to compute

$$\bar{h}_n = \text{argmax}_{h\in L^1} \frac{\left\langle K^\alpha_{\bar{t}_n} * g_n, h\right\rangle}{\|h\|_{L^1}}, \text{ via}$$

$$h_\tau = \frac{K^\alpha_{\bar{t}} * g}{\|h\|_{L^1}} - \frac{\left\langle K^\alpha_{\bar{t}} * g, h\right\rangle}{\|h\|^2_{L^1}} \frac{h}{|h|}.$$

Minimizing $\mathcal{J}_e,\ p < \infty$

$$\mathcal{J}_e(u) = |u|_{BV} + \lambda \|f - u\|_{\dot{B}^{\alpha-2}_{p,\infty}}$$
$$= \int_\Omega |\nabla u| + \lambda \sup_{t>0} \|H_t^\alpha * (f-u)\|_{L^p},$$

where $H_t^\alpha = t^{2-\alpha} P_t$ or $H_t^\alpha = t^{1-\alpha/2} W_t$.
Suppose $(u_n, \bar{t}_n)$ is known. Compute (u_{n+1}, t_{n+1}) via

- $\left(\frac{\partial \mathcal{J}_e}{\partial u} = \right),\ u_\tau = \nabla \cdot \left(\frac{\nabla u}{|\nabla u|}\right) +$
$\lambda \left\| H^\alpha_{\bar{t}_n} * (f-u) \right\|^{1-p}_{L^p} H^\alpha_{\bar{t}_n} * \left(|H^\alpha_{\bar{t}_n} * (f-u)|^{p-2} H^\alpha_{\bar{t}_n} * (f-u) \right).$
- $t_{n+1} = \text{argmax}_{t \in \{t_{k-1}, t_k = \bar{t}_n, t_{k+1}\}} \|H_t^\alpha * (f - u_{n+1})\|_{L^p}.$

Minimizing $\mathcal{J}_e,\ p = \infty$

$$\mathcal{J}_e(u) = |u|_{BV} + \lambda \|f - u\|_{\dot{B}^{\alpha-2}_{\infty,\infty}}$$
$$= \int_\Omega |\nabla u| + \lambda \sup_{t>0, h\in L^1} \frac{\langle H_t^\alpha * (f-u), h\rangle}{\|h\|_{L^1}}.$$

• **Algorithm:** The steps are the same as in the previous case, but now at each iteration we need to compute

$$\bar{h}_n = \text{argmax}_{h\in L^1} \frac{\left\langle H_{\bar{t}_n}^\alpha * (f-u_n), h\right\rangle}{\|h\|_{L^1}}, \text{ via}$$

$$h_\tau = \frac{K_{\bar{t}}^\alpha * (f-u)}{\|h\|_{L^1}} - \frac{\left\langle K_{\bar{t}}^\alpha * (f-u), h\right\rangle}{\|h\|_{L^1}^2} \frac{h}{|h|}.$$

Barbara

f

Using $\mathcal{J}_a$ with $u \in BV$, $v \in B^{-0.5}_{1,\infty}$

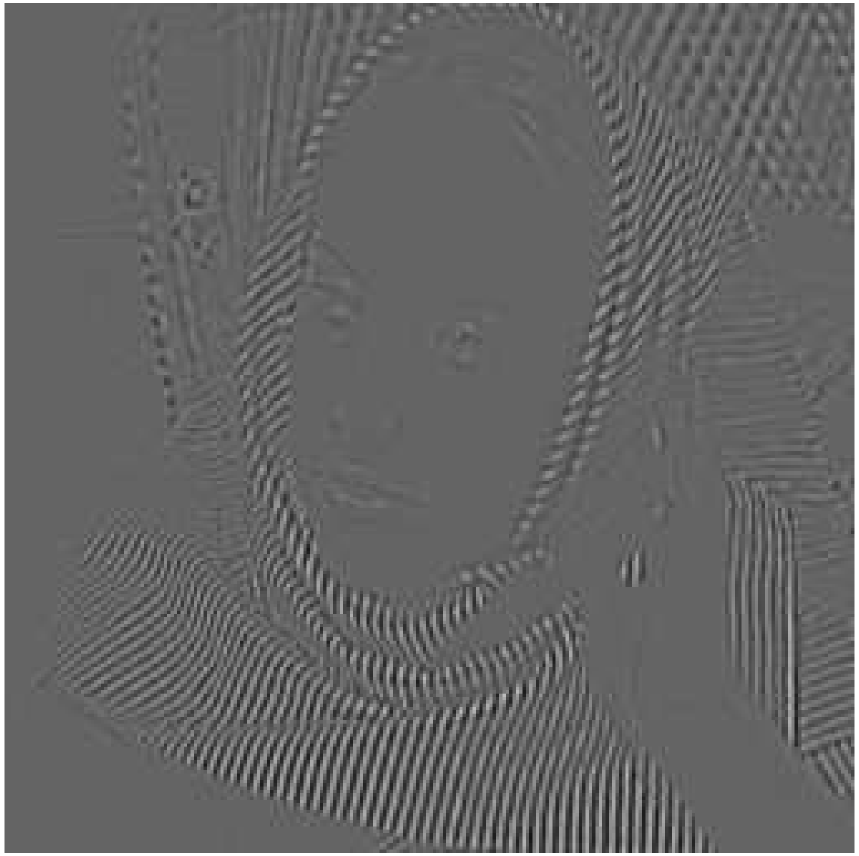
u | $f - u + 100$

Parameters: $\alpha = 1.5$, $p = 1$, $\mu = 1$, and $\lambda = 1e - 04$.

Using $\mathcal{J}_a$ with $u \in BV$, $v \in B^{-1}_{1,\infty}$

u

$f-u+100$

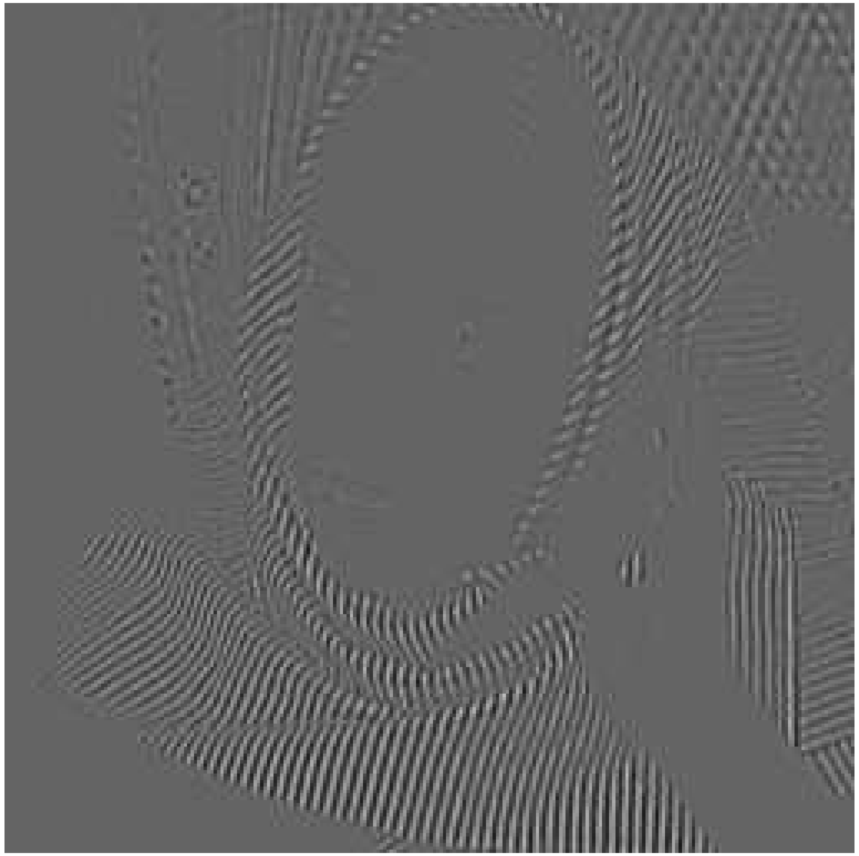


Parameters: $\alpha = 1.0$, $p = 1$, $\mu = 1$, and $\lambda = 3e - 03$.

Using $\mathcal{J}_a$ with $u \in BV$, $v \in B^{-1.5}_{1,\infty}$

u

$f-u+100$

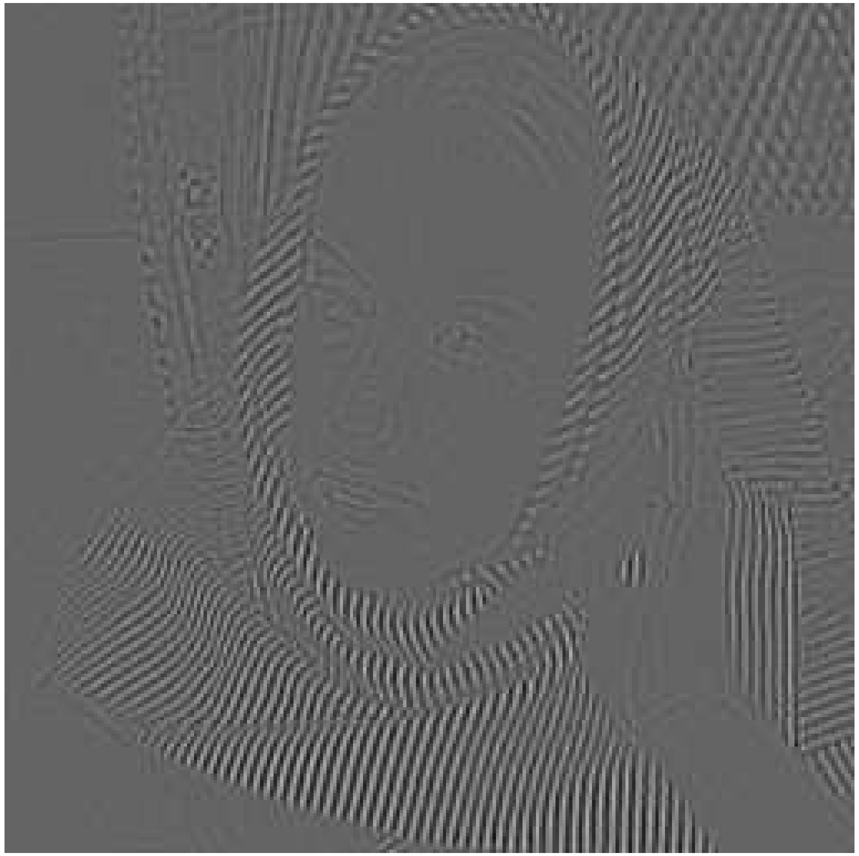


Parameters: $\alpha = 0.5$, $p = 1$, $\mu = 1$, and $\lambda = 0.5$.

Using $\mathcal{J}_a$ with $u \in BV$, $v \in B^{-1}_{\infty,\infty}$

u

$f-u+100$



Parameters: $\alpha = 1$, $p = \infty$, $\mu = 10$, and $\lambda = 1$.

Using $\mathcal{J}_e$ with $u \in BV$, $v \in B^{-1}_{1,\infty}$

u $\qquad$ $f-u+100$

Parameters: $\alpha = 1$, $p = 1$, $\lambda = 1500$.

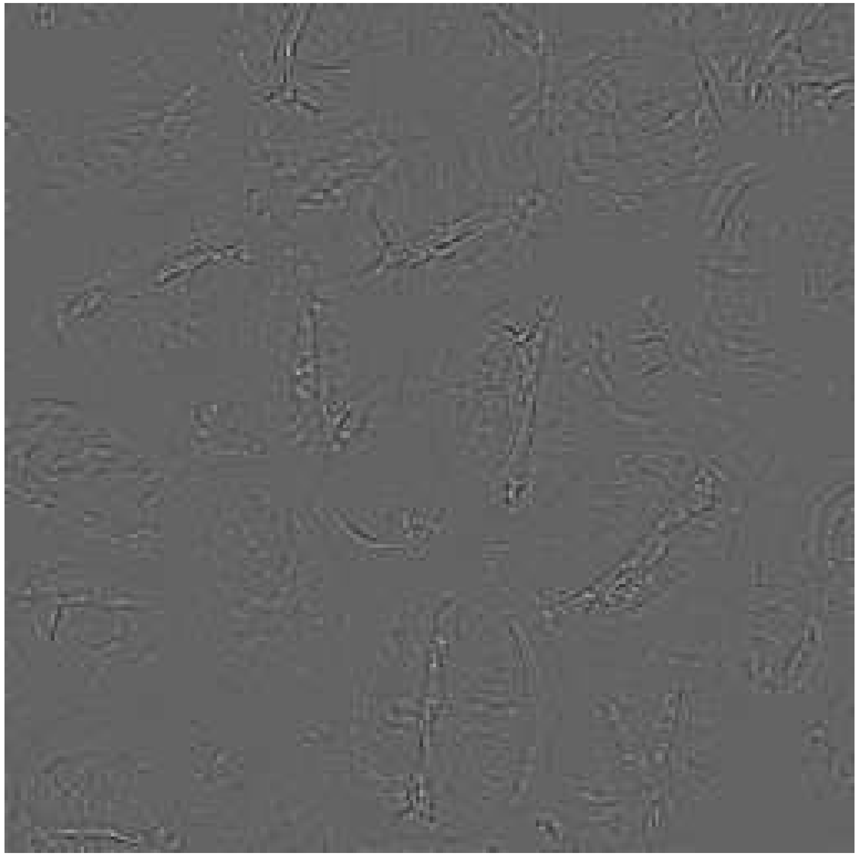
Coffee beans

f

Using $\mathcal{J}_e$ with $u \in BV$, $v \in B^{-1}_{1,\infty}$

u

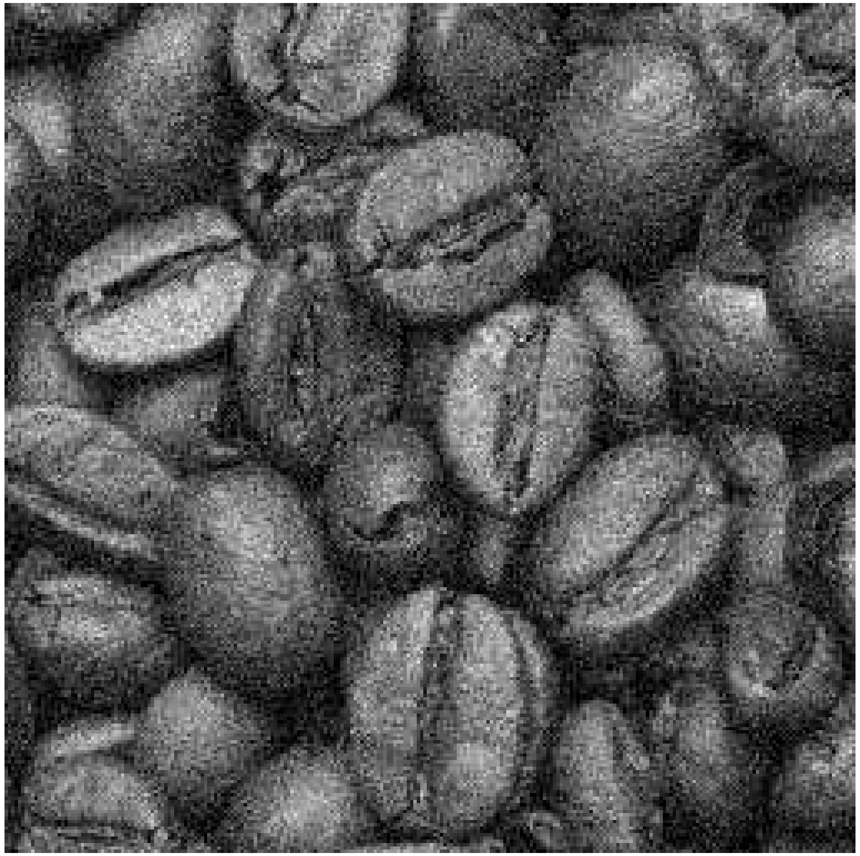
$f - u + 100$



Parameters: $\alpha = 1$, $p = 1$, $\lambda = 1500$.

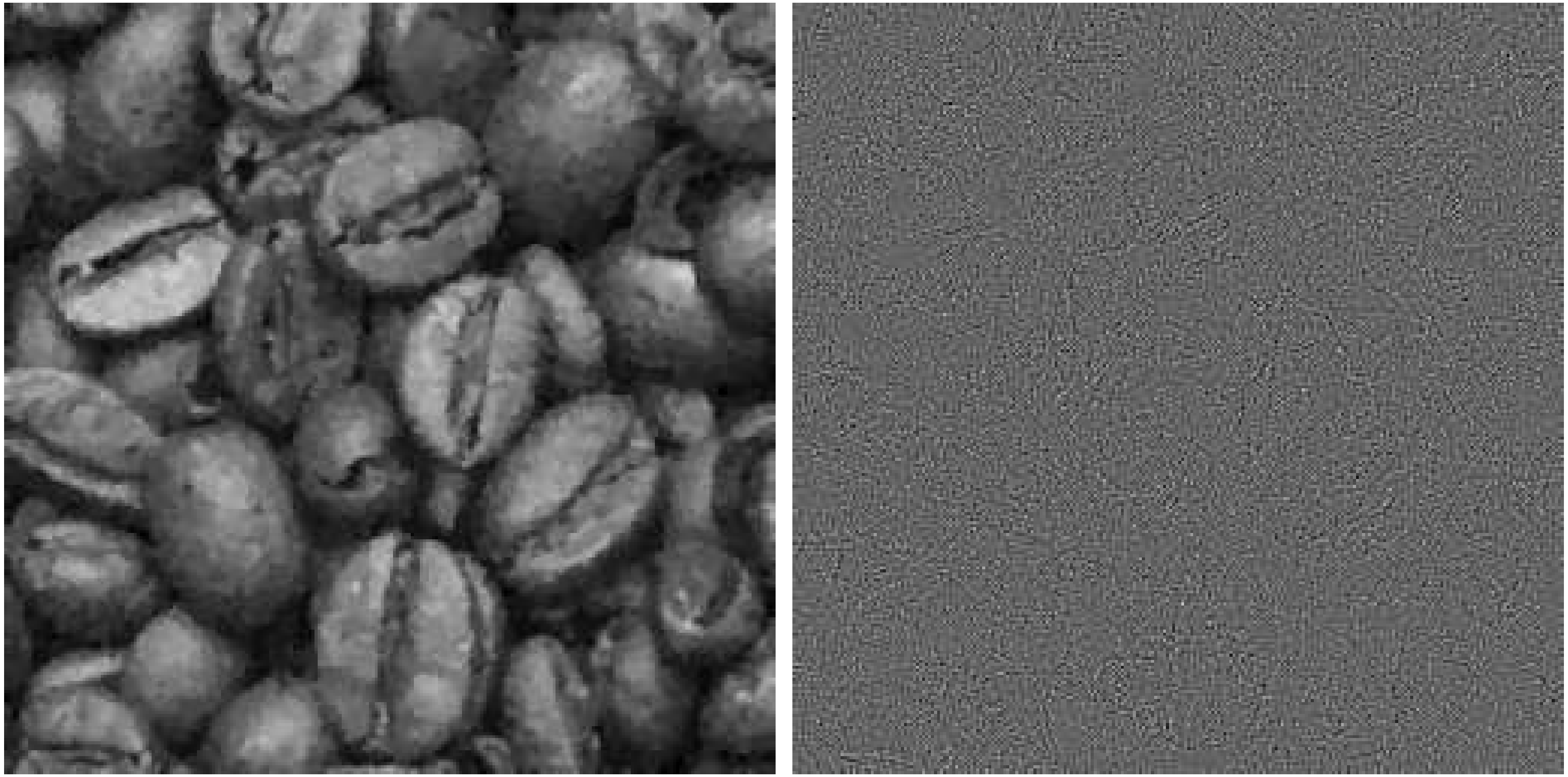
Coffee beans with additive Gaussian noise

f



Using $\mathcal{J}_e$ with $u \in BV$, $v \in B^{-1}_{1,\infty}$

u $\qquad$ $f - u + 100$



Parameters: $\alpha = 1$, $p = 1$, $\lambda = 2500$.

Stair-casing effects of total variation

The regularization term $\int |\nabla u| \, dx$ creates stair-casing effects in u in regions where f is very oscillatory.
To overcome this, One could replace $\int |\nabla u| \, dx$ with $\int \varphi(\nabla u) \, dx$, where

- $\varphi(z) \geq 0$ is convex (strictly convex near 0),
- lower semicontinuous, increasing and has linear growth at ∞.

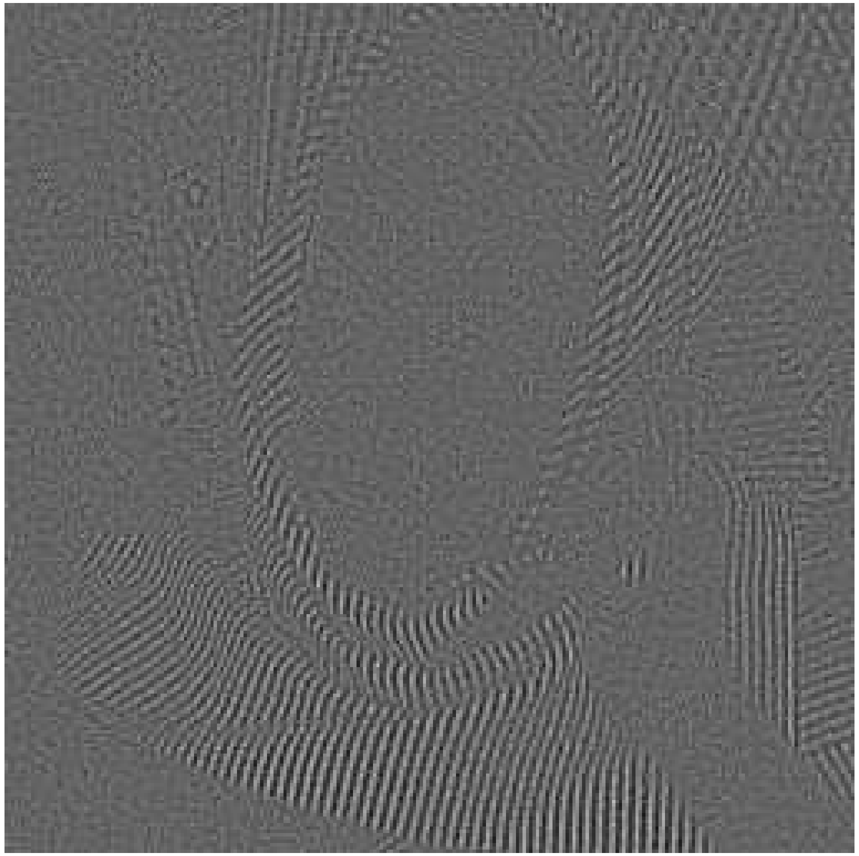
Related work:

- G. Aubert and L. Vese.
- P. Schultz, E. Bollt, R. Chartrand, s. Esedoḡlu, K. Vixie.
- S. Levine.
- among others.

Barbara with Gaussian noise

f

Using $\mathcal{J}_a$ with $v \in B^{-1}_{2,\infty}$



- $\varphi(\nabla u) = -\beta + \sqrt{|\nabla u|^2 + \beta^2}$, $\beta = \sqrt{10}$,
- $\alpha = 1$, $p = 2$, $\mu = 10$, and $\lambda = 0.01$.

Conclusion

- In these models, inteads of imposing $\|v\|_{L^p}$ on the oscillatory component v, we impose $\|K_t * v\|_{L^p}$ for some $t > 0$, where K_t is a smoothing kernel.
- Use $p = 1$ for texture (repeated patterns) decomposition. M. Green also shows that texture-like natural images when being convolved with a kernel of zero mean has a laplacian probability distribution.
- Use $p = \infty$ for cases where one wants to capture more oscillations including non repeated patterns.

Thank You!

This presentation consists of materials from these papers:

1. T. Le and L. Vese, *Image decomposition using total variation and div*(bmo), Multiscale Modeling and Simulation, SIAM Interdisciplinary Journal, vol.4, num. 2, pp. 390-423, June 2005.
2. J. Garnett, T. Le, and L. Vese, *Image decompositions using bounded variation and homogeneous Besov spaces*, UCLA CAM Report 05-57, Oct. 2005.